ON THE MODELING OF TIME-VARYING DELAYS

A Thesis

by

CHIRAG SHAH

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

May 2004

Major Subject: Mechanical Engineering

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ABSTRACT

On the Modeling of Time-varying Delays. (May 2004)

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This thesis is an effort to develop generalized dynamic models for systems with time-varying time delays. Unlike the simple time-delay model characterized by a transportation lag in the case of a fixed time delay, time-varying delays exhibit quite different characteristics, making the development of easy to use models a difficult endeavor.

First an algorithm is developed to predict the actual input-output behavior when the input signal is directly fed into a device that characterizes the time-varying delay. Input-output behaviour generated with this algorithm serves as the truth model for subsequent approximate model development. Simulation results for different classes of delay and different inputs were obtained using the truth model. The input functions were limited to steps, ramps and sinusoids. This limited class of inputs and delays defines the scope of this thesis and the results are to be interpreted as such.

The methodology adopted to identify the basic underpinnings of models was system identification where input-output data came from the truth model. Models for the aforementioned classes of inputs and delays were then derived using elementary system identification tools. These models were then carefully analyzed to extract trends by changing the delay parameter. A satisfactory trend was observed in the case of linearly varying time delay. A generalized model for the linearly varying time delay with step and polynomial inputs was developed. An attempt was also made at developing a generalized model for sinusoidally time-varying time delays.

This study proposes a model for linearly time-varying time delay, whose structure is not surprisingly also dependent on the class of inputs. It is shown that the derived model reduces to the well known model in the case of a fixed delay.

To my loving parents.

ACKNOWLEDGMENTS

I would like to express my sincere gratitude to my advisor, Professor Suhada Jayasuriya, without his guidance this thesis would not have been accomplished. I appreciate his time and efforts throughout the duration of my research.

I wish to thank Professor Won-jong Kim and Professor Shankar Bhattacharyya for serving on my thesis committee.

I'm fortunate to have had the company of a talented and motivated bunch of colleagues in the Mechanical Engineering Controls Lab, here at Texas A&M University. I would like to thank Aninda, Ankush, Avinash, Gautam, Sumit and Suren for their time and constructive suggestions.

My wholehearted gratitude goes to my parents, Bhavna and Laxmikant Shah, for their immeasurable support throughout my life. Without their love, encouragement and support, I would have never come this far.

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CHAPTER I

INTRODUCTION

Time delays often arise in control systems, both from delays in the process itself and from delays in the processing of sensed signals. Process industries often have processes with time delays introduced due to the finite time it takes for material to flow through pipes. In measuring altitude of a spacecraft, there is a significant time delay before the sensed signal arrives back on Earth. A recent example of it is interplanetary telecommunication through Mars rover. In modern digital control systems, time delay can arise from sampling, due to cycle time of the computer and the fact that data is processed at discrete intervals. Thus, time delay could be due to heat and mass transfer in chemical industries, heavy computations and hardware restrictions in computational systems, high inertia in systems with heavy machinery and communications lag in space craft and remote operation. Chemical processing systems, transportation systems, communication systems and power systems are typical examples that exhibit time-delays. The effect of the time delay on the system dynamics, however, depends on the delay and the system characteristics [1], [2].

A pure time delay, an essential element in the description of these systems, has the property that input and output are identical in form, but output is shifted along the time axis [3].

Time delays always reduce the stability of systems. The control action cannot be realized immediately because of the time delay. This can lead to instability of a system. To illustrate this consider a time delay system with a PID Controller. Assume that a set-point change is just being made and the PID controller is working

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to bring the process variable to the desired value. During the interval of the process dead time, the process variable has yet to react to the set-point change and therefore no dynamical information is available for the derivative part to make the correct prediction. Therefore, if it is being used, the derivative part will result in unnecessary oscillations in the system. Hence it destabilizes the system. Hence, if a time-delay process is controlled by a PID Controller, then the derivative part of the controller is always turned off. But when the derivative part is removed, the future control errors cannot be predicted by the controller, which is necessary for time-delay systems [2]. Therefore, it is important that time delay systems are well studied, analyzed and modelled before control is attempted.

In steel rolling process, the measurement point is located at some distance downstream of the steel press. Consequently, there is always a time delay in feeding back the plate thickness to the press controller.

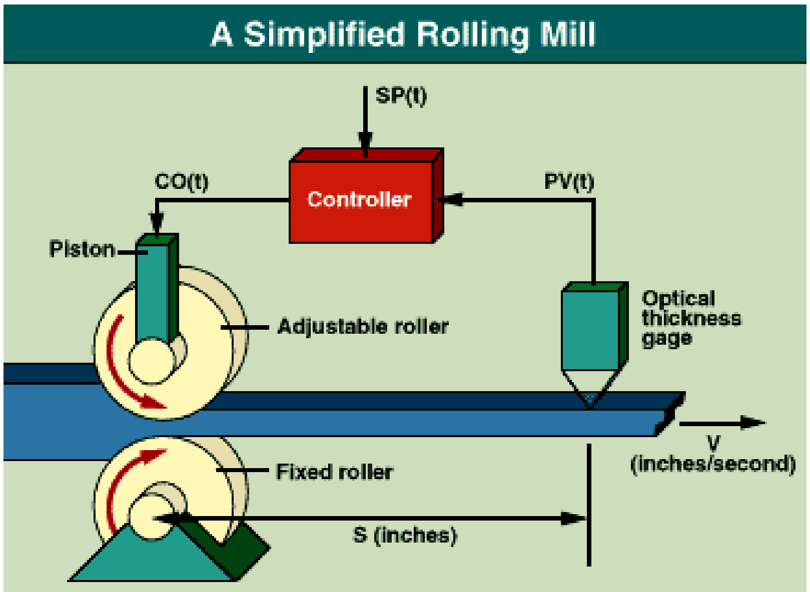

Fig. 1. Simplified rolling mill (Reprinted from [4]).

Consider, for example, the rolling mill shown in figure 1, which produces a continuous sheet of some material at a throughput of V inches per second. A feedback controller uses a piston to modify the gap between a pair of reducing rollers that squeezes the material into the desired thickness. The time delay in this process is caused by the separation S between the rollers and the thickness gauge [4].

As another example, consider the problem of controlling the speed of a steam engine running an electric power generator under varying load conditions. A control system for this purpose is the centrifugal governor. This control system consists of a set of fly balls or rotating weights suspended from levers which are connected to the steam valve. The fly balls are driven by the engine. An increase in the engine speed would result in a larger centrifugal force which would raise the fly balls and thereby would reduce the steam flow into the engine causing a reduction in its speed. However, there would be a delay between the time that engine speed increased and the time that steam flow is reduced due to inertial of the control equipment [5].

A thermal plant usually consist of various pieces of heat transfer apparatus connected together by pipes. Simulation of transport delay introduced by these pipes is frequently required to calculate system dynamics. When the fluid velocity through the pipe is constant, the transfer function of the piping lag is easily derived. When the velocity of the fluid is a function of time, delay introduced in the system also becomes function of time. And it is not possible to find a transfer function in the usual way. This has been a source of difficulty to many investigators in the past [6].

So, time delay systems present a wide range of challenges in implementing controllers for them.

A. Previous work

Time delays fall into two main categories:

1. Fixed time delay

2. Time-varying delay

1. Fixed delay

Delays, which remain constant with time are called fixed time delays. Figure 2 shows

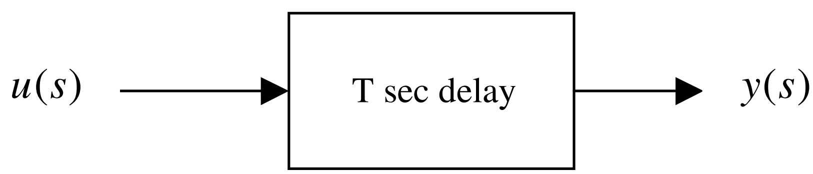

Fig. 2. Delay block.

a fixed time delay of T sec. The Laplace Transform of the system output is

$$y(s) = e^{-st}u(s),$$

where,

$$e^{-st} = 1 - s + \frac{s^2}{2} - \frac{s^3}{3!} + ...$$

In the time domain, we can write it as

$$y(t) = u(t - T). \tag{1.1}$$

A substantial work has been done in the past on the approximation of the constant delay [7], [8]. Many equivalent frequency domain Transfer Functions have been proposed to describe constant time delay systems. The methods that are employed are closely related to the Padé approximation. A Padé approximant is the ratio of two polynomials constructed from the coefficients of the Taylor series expansion of a function [9].

$$e^{-s\tau} = \frac{1 - s(\frac{\tau}{2}) + s(\frac{\tau}{2})^2 - ...}{1 + s(\frac{\tau}{2}) + s(\frac{\tau}{2})^2 + ...}$$

Early work was due to Thomson (1949, 1952) who investigated the approximation of delay with maximally flat frequency response around $\omega = 0$ [10], [11]. Later, Storch (1954) investigated the synthesis of constant time-delay networks using Bessel polynomials [12]. Lam (1988) has worked on the problem of approximating $e^{-s\tau}$ by Padé approximants with numerator and denominator polynomials of the same degree [13].

Another approximant for fixed time delay is given by Laguerre formula as [14]

$$L_n(sT) = \left(\frac{1 - \frac{sT}{2n}}{1 + \frac{sT}{2n}} \right)^n,$$

where $n = \{1, 2, ...\}$, the set of natural numbers. This approximation is used when the order of the Padé approximation is large.

Yet, another approach consists of regarding $e^{-s\tau}$ as $1/e^{s\tau}$ and replacing $e^{s\tau}$ by its Taylor series expansion, which can be written as [9]:

$$E1 = \frac{1}{(1 + sT)},$$

$$E2 = \frac{1}{(1 + sT + T^2 s^2/2!)}.$$

However, these approximations do not apply in the case of time-varying time delay.

2. Time-varying delay

Delays, which are functions of time, are called time-varying delays. Linear systems with time-varying delays may be represented as

$$
\begin{aligned}
\dot{x}(t) &= (A + \triangle A(t))x(t) + (A_{d1} + \triangle A_{d1}(t))x(t - d_1(t)) \\
&\quad + (B + \triangle B(t))x(t) + (B_{d1} + \triangle B_{d1}(t))x(t - d_1(t)) \qquad (1.2) \\
y(t) &= Cx(t) \qquad (1.3)
\end{aligned}
$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control,$y(t) \in R^q$ is the measurement output, $d_i(t) \in R$, i = 1, 2, ... are time-varying time delays with the following assumptions:

$$0 \leq d_i(t) < \infty, \quad \dot{d}_i(t) \leq m_i < 1, \quad i = 1, 2, ...$$

and all the matrices have appropriate dimensions [15].

Time-varying delay systems show significantly different characteristics from that of fixed time delay systems. Satisfactory modelling of time-varying delay is important for the synthesis of effective control systems for such systems. Professor G. A. Korn has pointed out some of the difficulties inherent in the simulation of variable time delays [16]. Nonetheless at the present time such dynamic models are not readily available in the literature. Vichnevetsky [17] tried to extrapolate conventional methods of approximation of a constant delay to time-dependent delays by making the coefficient time-dependent as well. Seddon [18] investigated a scheme for converting a problem with variable delay to one with fixed delay. Leonard [9] compared various approximants available for fixed delay. Robinson and Soundack presented a method for the identification of time delays and parameters in linear systems [19].

In the literature properties such as performance, stability and control with fixed time delay have been extensively studied [5], [20], [21], [22]. However, the time-varying delay problem has not received the same level of attention [18], [23], [24]. Consequently, there is a definite need for identification and modeling of time-varying delay in systems. This thesis is an effort in that direction.

B. Research, objective and organization of thesis

The main objective of this research is to develop a generalized dynamic model for time-varying delay in systems.

Layout of the thesis is as follows. Chapter II describes the methodology followed to obtain the generalized dynamic models. Validation of the method is also presented in this chapter. In Chapter III, various approximation models proposed for time-varying delay are discussed and are supported with simulations. The work is summarized and conclusions are made in Chapter IV.

CHAPTER II

METHODOLOGY

In this Chapter, the method followed to obtain the approximations for time-varying delay is discussed. Validation of the method is performed by comparing the results with those for a constant delay.

Satisfactory approximations can be achieved by generating input-output behaviour of systems with time-varying time delay. The input functions considered in this research are step, ramp, polynomials and sinusoids and included are linearly varying time delay and sinusoidally varying time delays.

In case of time-varying time delay systems, process dynamics are given by

$$y(t + \tau(t)) = u(t). \tag{2.1}$$

The generated input function and the delay function are applied to equation (2.1) to obtain the output.

This serves as the truth model for obtaining input-output behaviour of systems with time-varying time delay. Once the true output is obtained, system identification with these simulated results is carried out to obtain a model approximation for several cases. System identification provides tools for developing mathematical models of dynamic systems based on observed input-output data [25], [26]. Data obtained from the simulated results are fed into an ARX model structure, which uses the least square estimation method. Data is fed into the system ID toolbox as an Id-data object that contains the input-output data sequences. Model structures with different orders are obtained and compared and the best order is chosen. These models are validated by comparing them with the true outputs. In this way models for different classes of inputs are obtained. These models are then analyzed carefully to observe trends by changing the delay parameter.

A. Classification of time-varying delays

Time-varying time delays can be classified into many categories. Some of these are,

- linearly time-varying delay,
- sinusoidal time-varying delay,
- randomly time-varying delay,
- network delay, and
- distributed delay.

Linearly time-varying delays and sinusoidal time-varying delays are considered in this research.

B. Time-varying delay representation

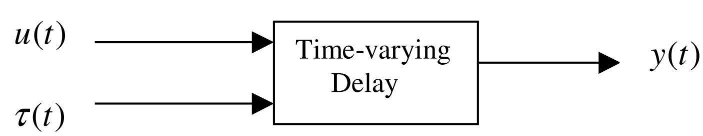

Fig. 3. Time-varying delay block.

The system with time-varying delay shown in figure 3 can be represented as shown in equation (2.1) as

$$y(t + \tau(t)) = u(t), \tag{2.2}$$

where $\tau(t)$ is the time-varying delay, $u(t)$ is the control input and $y(t)$ is the output.

It is interesting to note that for a fixed time delay system we use the representation

$$y(t) = u(t - \tau), \tag{2.3}$$

where τ is a fixed delay which remains same irrespective of observation time. But for time-varying system, $\tau(t)$ is the time delay at time t and its value is defined according to the delay function. So for different instants of time, the value of the time delay may be different. Hence, we cannot apply the same representation in the case of time-varying time delay.

To understand this, consider a transport facility in which the information is "written" on a medium such as magnetic tape and "read" at a second location. The medium carries the information at some velocity from the writing point to the reading point. If the "length of path" L and the velocity of the medium V are fixed, then certainly the delay $\tau(t)$ is fixed given by [18],

$$\tau = \frac{L}{v}. \tag{2.4}$$

In this case, the system output can be written as in equation (2.3). If the position of the read head is varied linearly, making delay an increasing function of time, then the delay is given by,

$$\tau(t) = \frac{L(t)}{v}. \tag{2.5}$$

Thus the information that is written on the "write head" at time t will be read by "read head" at time (t+$\tau(t)$). To express it mathematically, it can be written as

$$y(t + \tau(t)) = u(t). \tag{2.6}$$

Thus, systems with time-varying time delay, should follow equation (2.1) for its representation.

C. SIMULINK variable transport delay block

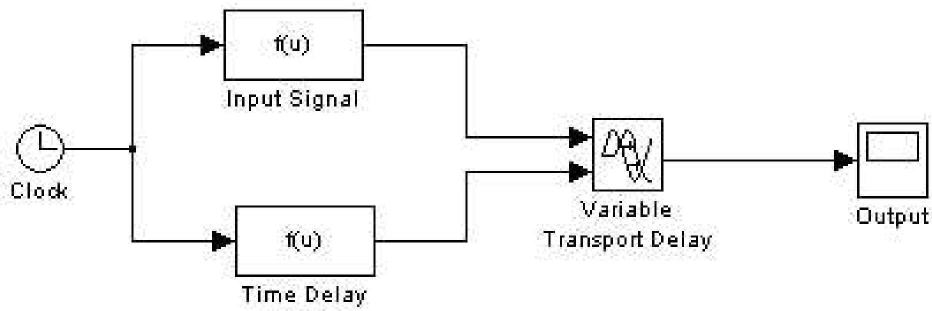

Fig. 4. Variable transport delay block in SIMULINK.

MATLAB has a "Variable Transport Delay" block, shown in figure 4 is used to simulate variable time delay. The block accepts two inputs: the first input is the signal that passes through the block; the second input is the time delay. The block outputs the signal at the time that corresponds to the current simulation time minus the delay time. It uses the representation

$$y(t) = u(t - \tau(t)). \tag{2.7}$$

As explained earlier, this representation is not valid for time-varying delay. This is a simple transportation lag equation which is valid only for fixed delay. Hence an algorithm is developed to obtain the actual input-output behaviour of the systems with time-varying delays. This algorithm serves as the truth model for system identification.

D. Truth model validation

To verify the veracity of the truth model results, limit analysis is performed to recover well known results for the fixed time delays. A constant delay function is fed to the truth model to obtain the results. Following example shows the output of the truth model for constant delay.

Consider a ramp input to the block. Here,

$$u(t) = t,$$

$$\tau = 1.$$

These functions are plugged into the equation (2.1) to obtain the output.

$$y(t + 1) = u(t)$$

The response of the truth model for first 10 sec is shown in figure 5.

In another example, constant input function is considered as shown in equation (2.8).

$$u(t) = 1, \tag{2.8}$$

$$\tau(t) = 1. \tag{2.9}$$

This input is subjected to a constant delay of 1 second. The output obtained using the truth model is shown in figure 6. These graphs shows that the methodology followed to obtain results for systems with fixed delay is valid.

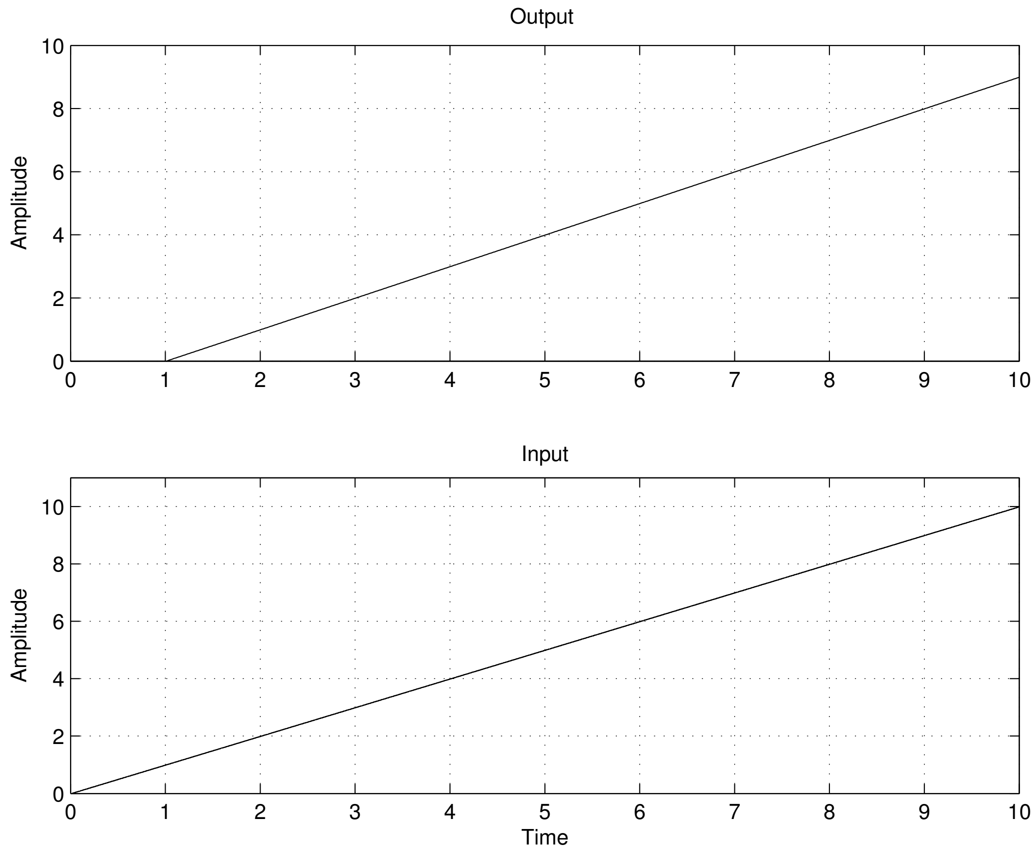

Fig. 5. Input-output response of the truth model for the ramp input with fixed delay.

E. ARX model

The equations for the ARX structure are obtained from *System Identification* by Lung [26]. In ARX (autoregressive exogenous) model, a dynamical system with input signal $u(t)$ and output signal $y(t)$ sampled in discrete time $t = 1, 2, 3...$ is considered. It is assumed that these signals are related through a difference equation.

$$y(t) + a_1 y(t-1) + ... + a_n y(t-n) = b_1 u(t-1) + ... + b_m u(t-m). \tag{2.10}$$

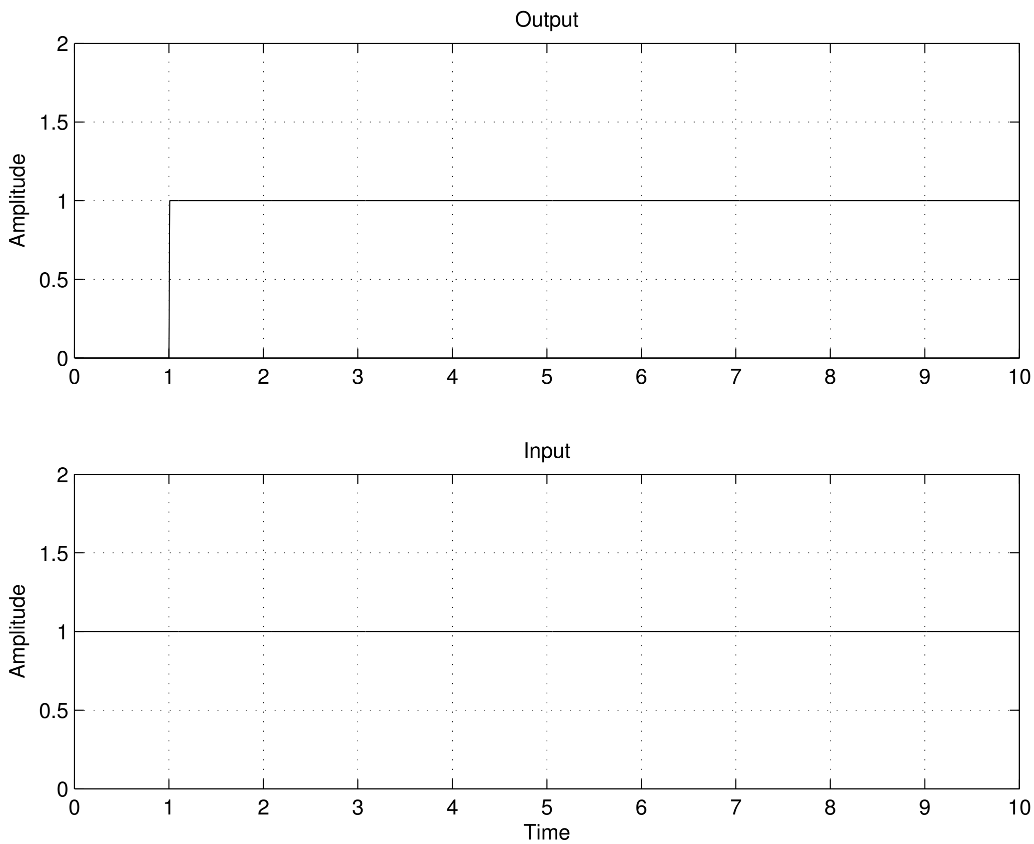

Fig. 6. Input-output response of the truth model for the step input with fixed delay.

A pragmatic and useful way to see equation (2.10) is to view it as a way of determining the next output value given previous observations:

$$y(t) = -a_1 y(t-1) - ... - a_n y(t-n) + b_1 u(t-1) + ... + b_m u(t-m). \tag{2.11}$$

For more compact notation we introduce the vectors:

$$\theta = [a_1 ... a_n \quad b_1 ... b_m]^T, \tag{2.12}$$

$$\varphi(t) = [-y(t-1) ... -y(t-n) \quad u(t-1) ... u(t-m)]^T. \tag{2.13}$$

With these, equation (2.11) can be written as

$$y(t) = \varphi^T(t)\theta. \tag{2.14}$$

To emphasize that the calculation of $y(t)$ from past data in equation (2.11) indeed depends on the parameters in θ, we shall rather call this calculated value $\hat{y}(t|\theta)$ and write

$$\hat{y}(t|\theta) = \varphi^T(t)\theta. \tag{2.15}$$

F. Time derivatives consideration

For time-varying delay systems, it has been shown that the stability is dependent on the maximum value of the first derivative of the time-varying delay [21]. It has been observed that when the first derivative of the delay function is greater than 1, it causes significant degradation of the control system performance, stability and robustness.

Hence for system stability and better performance, the following restriction has been employed [15], [24]:

$$\frac{d[\tau(t)]}{dt} < 1. \tag{2.16}$$

This factor is taken into consideration in this research for the purpose of model building.

CHAPTER III

APPROXIMATION MODELS DEVELOPED FOR TIME-VARYING TIME DELAYS

This chapter describes the various approximation models proposed for linearly time varying delays and sinusoidal time-varying delays. Simulation results and examples are provided to support the results. Approximation within the context of this study means the determination of the parameters of input and delay function in such a way that a different function is characterized in a reasonably accurate manner over a specified range of the variable. The purpose of these approximations is to reduce the difficulties in theoretical and numerical analysis [27].

A. Linearly time-varying delays

Linearly time-varying delay is analyzed for different kinds of inputs. Linearly time-varying delay is defined as

$$\tau(t) = kt, \tag{3.1}$$

where $k \in [0, 1]$ and is constant. A restriction has been imposed as $k < 1$ so that the first derivative of the delay is always less than one.

The truth model output is obtained for step, ramp, polynomial and sinusoidal input functions that are subjected to linearly time-varying delay. These input-output data are then fed into the system identification toolbox to obtain the relation between data. After careful analysis and a number of observations from numerous simulations, some approximation models are proposed. Since approximation obtained for step and ramp inputs are special cases of approximation obtained for polynomial inputs, we shall consider polynomial inputs.

1. Polynomial input

A polynomial input is defined as

$$u(t) = at^n + bt^{(n-1)} + ... + ct + d,$$

where a, b, c, d are arbitrary constant coefficients and n is the degree of the input polynomial. System equation with this input and delay function is

$$y(t + kt) = at^n + bt^{(n-1)} + ... + ct + d. \tag{3.2}$$

The known parameters of the input and delay functions are, the first derivative of the delay, degree of the input polynomial and coefficients of the polynomial. The input-output data were generated by the truth model by changing these parameters. System identification of the input-output data sequence were carried out to obtain the relation between the input and the output data. Table I shows some of the simple relations obtained between the input and the output functions in time domain.

Table I. Variation in the approximation model parameter with changes in the input and the delay function parameters.

Input	Delay	Appoximations
$t^2 + t$	$0.1t$	$\frac{y(t)}{u(t)} = 0.8677$
$2t^2 + t$	$0.1t$	$\frac{y(t)}{u(t)} = 0.8677$
$t^2 + t$	$0.2t$	$\frac{y(t)}{u(t)} = 0.7638$
$t^3 + t^2 + t$	$0.2t$	$\frac{y(t)}{u(t)} = 0.7021$
$t^2 + 2t$	$0.3t$	$\frac{y(t)}{u(t)} = 0.6804$
$0.1t^3 + 0.7t^2 + t$	$0.01t$	$\frac{y(t)}{u(t)} = 0.9803$
$2t^4 + 1.5t^2$	$0.5t$	$\frac{y(t)}{u(t)} = 0.3209$
$0.001t^5 + 0.005t^4 + 0.025t^3 + 0.02t$	$0.25t$	$\frac{y(t)}{u(t)} = 0.5132$

Some of the observations made from these calculations are

1. For higher coefficients of the delay function, the approximation value is lower.

2. Change in the coefficient of the input polynomial function, does not change the approximation value.

3. With higher degree polynomials, the approximation value is lower.

Based on these numerous calculations, an approximation is developed and verified with the truth model output. These results are also compared with the MATLAB SIMULINK block "Variable Transport Delay". This block delays the input by variable amount of time.

The obtained model is a relation between input $u(t)$ and the output $y(t)$. The system input-output behavior with linearly time varying delay subject to a polynomial input may be written as

$$\left[\left(\frac{1}{1+\frac{d\tau}{dt}}\right)^{n}+\left(\frac{1}{1+\frac{d\tau}{dt}}\right)^{(n-1)}+...+\left(\frac{1}{1+\frac{d\tau}{dt}}\right)\right]\frac{1}{C} \tag{3.3}$$

where C is a constant and equal to the number of terms in the input polynomial, n is the degree of the polynomial. In the case of a complete polynomial input, equation(3.3) can be written in geometric series form as

$$\left(\frac{1}{1+\frac{d\tau}{dt}}\right)\left[\frac{1-(\frac{1}{1+\frac{d\tau}{dt}})^{n}}{1-(\frac{1}{1+\frac{d\tau}{dt}})}\right]\frac{1}{C} \tag{3.4}$$

To understand the veracity of the approximation model, let us consider for instance a degree two polynomial. Output for the following polynomial input subjected to linearly time-varying delay is obtained through the truth model.

$$\begin{aligned} u(t) &= 2t+0.5t^{2}, \\ \tau(t) &= 0.1t. \end{aligned} \tag{3.5}$$

$$y(t+0.1t)=2t+0.5t^{2}. \tag{3.6}$$

Figure 7 shows input and output for the truth model.

For polynomial under consideration, according to the proposed approximation model, $n=1$, $C=2$ and $\frac{d\tau}{dt}=0.1$.

Hence,

$\left(\frac{1}{1+0.1}\right)^{1}=0.9091,$

$\left(\frac{1}{1+0.1}\right)^{2}=0.8264.$

$$\begin{aligned} \frac{y(t)}{u(t)} &= \frac{0.9091+0.8264}{2}, \\ \frac{y(t)}{u(t)} &= 0.8677. \end{aligned}$$

Figure 8 shows the comparison between the truth model output and the proposed

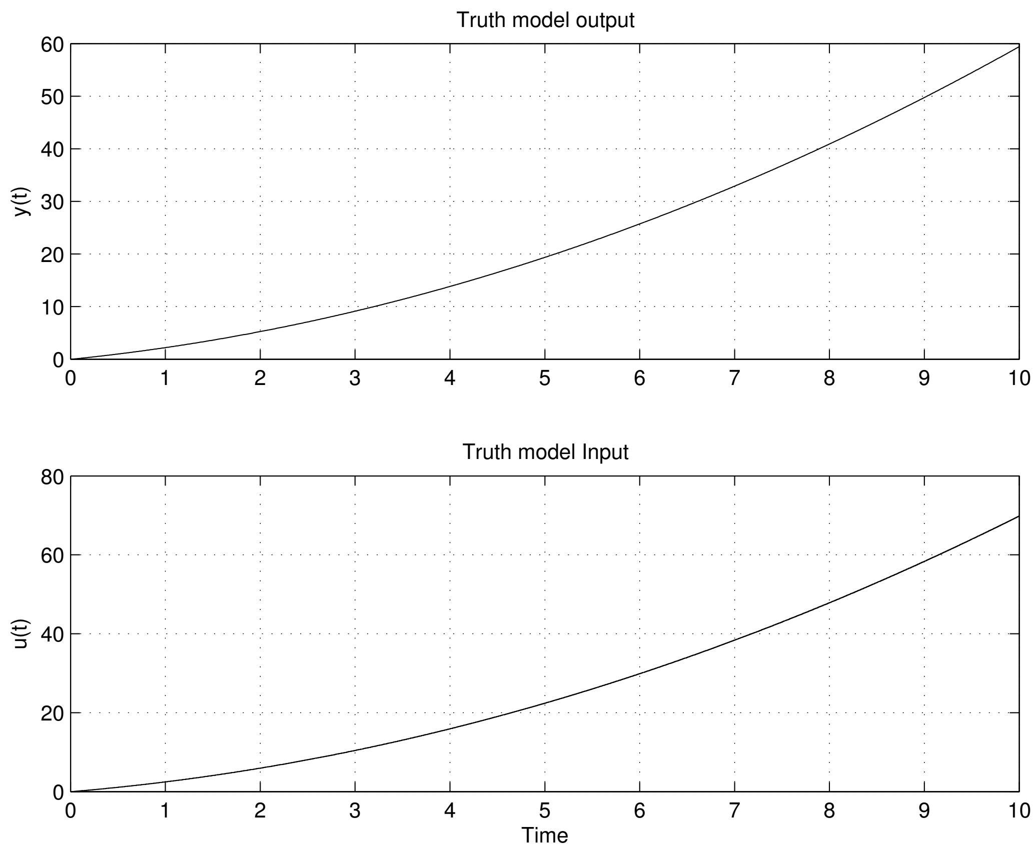

Fig. 7. Input-output behaviour for the truth model for the polynomial input with linearly time-varying delay.

model output for the polynomial in equation (3.6). Approximation error for polynomial input depends on the degree of the polynomial. For higher degree polynomials, approximation error is more. Figure 9 shows the % error between the truth model output and the proposed model output for the polynomial in equation (3.6).

Note that step and ramp inputs are special cases of the model proposed in equation (3.3). These cases are illustrated by the following three examples.

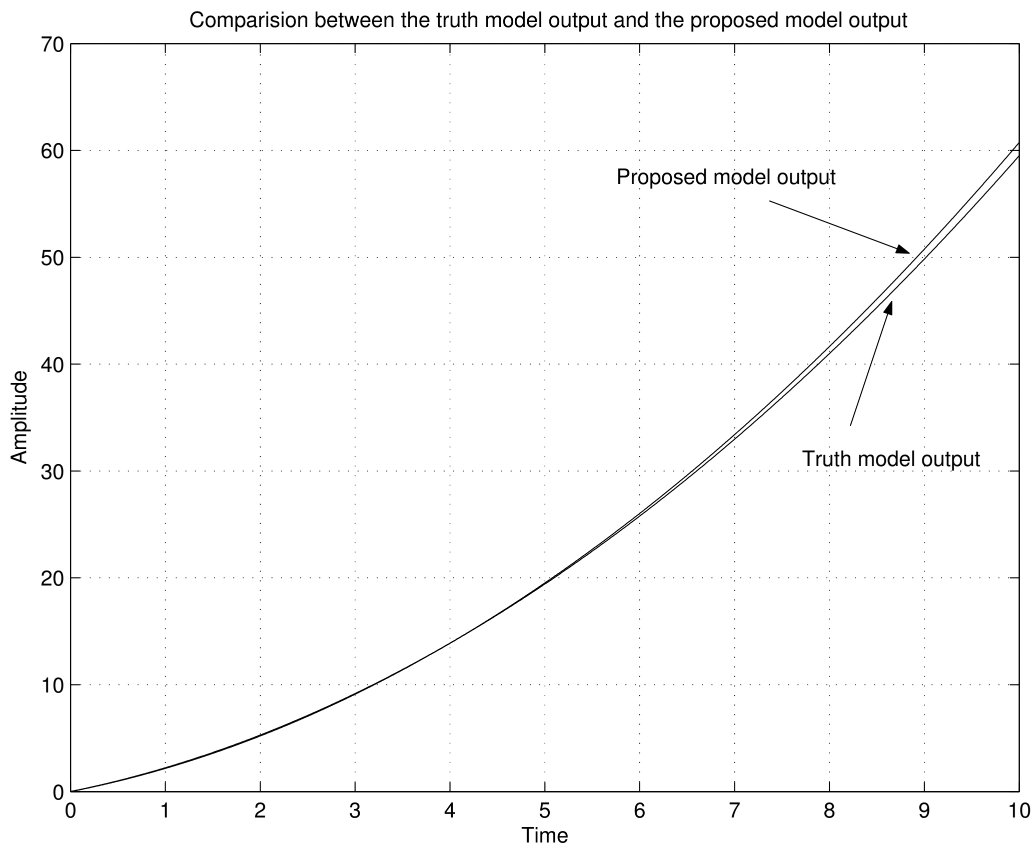

Fig. 8. Comparison between the truth model output and the proposed model output for the polynomial input with linearly time-varying delay.

Example 1:

A constant input is subjected to a linearly time-varying delay $\tau(t) = kt$. Constant input is a zero degree polynomial. According to the approximation model,

$$\frac{y(t)}{u(t)} = \left(\frac{1}{1+\frac{d\tau}{dt}}\right)^0,$$

$$\frac{y(t)}{u(t)} = 1.$$

Thus, when the input is constant, the output is the same as input.

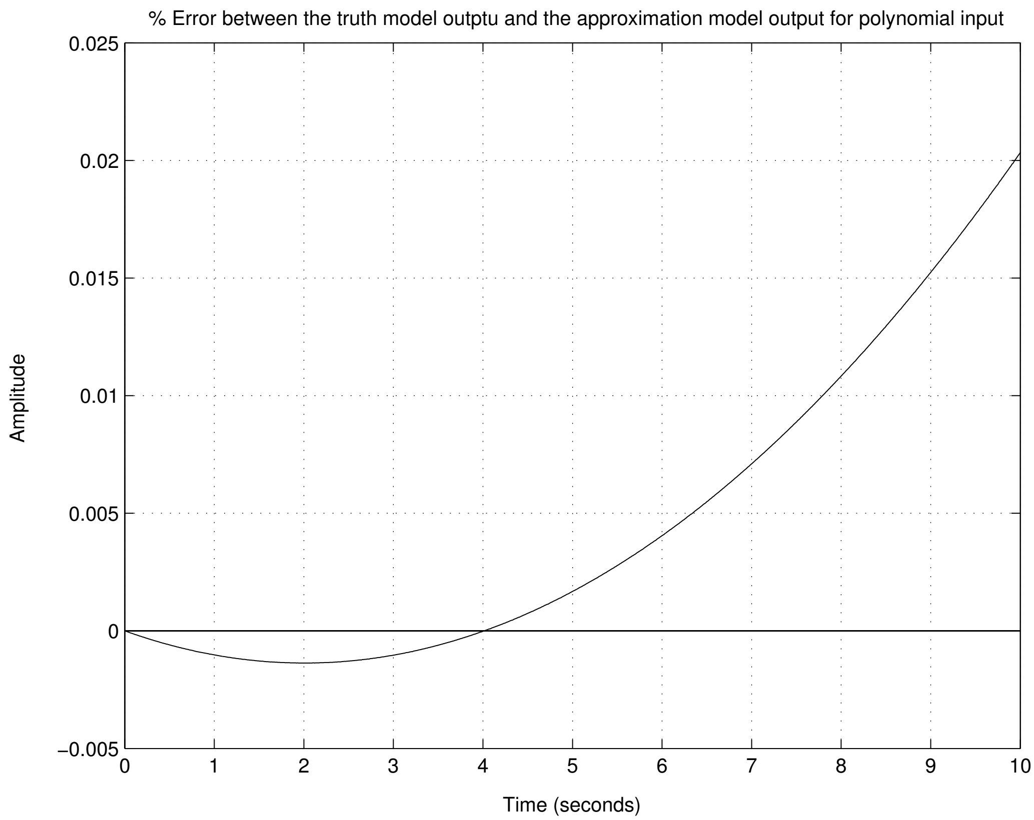

Fig. 9. Percentage error between the the truth model output and the proposed model output for the polynomial input with linearly time-varying delay.

Example 2:

Consider a system with no delay and subjected to a polynomial input

$$\begin{aligned} u(t) &= at^n + bt^m, \\ \tau(t) &= 0. \end{aligned}$$

When there is no delay in the system, $\frac{d\tau}{dt} = 0$.

$$\begin{aligned} \frac{y(t)}{u(t)} &= \left[\left(\frac{1}{1+0}\right)^n + \left(\frac{1}{1+0}\right)^m\right]\frac{1}{2}, \\ \frac{y(t)}{u(t)} &= 1. \end{aligned}$$

Hence the input-output behavior reduces to the well known no-delay response as expected. Thus with no delay in the system, the output is the same as input.

Example 3:

In this example, a ramp input is considered with linearly time-varying delay.

$$u(t) = at$$

$$\tau(t) = kt.$$

$$\frac{y(t)}{u(t)} = \left(\frac{1}{1 + \frac{d\tau}{dt}}\right)^{1}. \tag{3.7}$$

When the input is a ramp, the approximation model reduces to equation (3.7) .

2. SIMULINK block output

As explained earlier in this thesis, MATLAB has a "Variable Transport Delay" block which is used to simulate variable time delay. The truth model output and the SIMULINK block output are compared as shown in figure 10. The difference in the two outputs shows that there is an error because of equation (2.7). Figure 11 shows the % error between the the truth model output and the SIMULINK block output for polynomial input.

3. Sinusoidal input

Sinusoidal inputs with linearly time-varying delay are considered for model development. Simulation for various sine inputs and linear delays is carried out. It is observed that the model parameters depend on the frequency of the input signal and the derivative of time delay function. Delay brings additional frequency components in the output function. Obtaining the generalized approximation model has

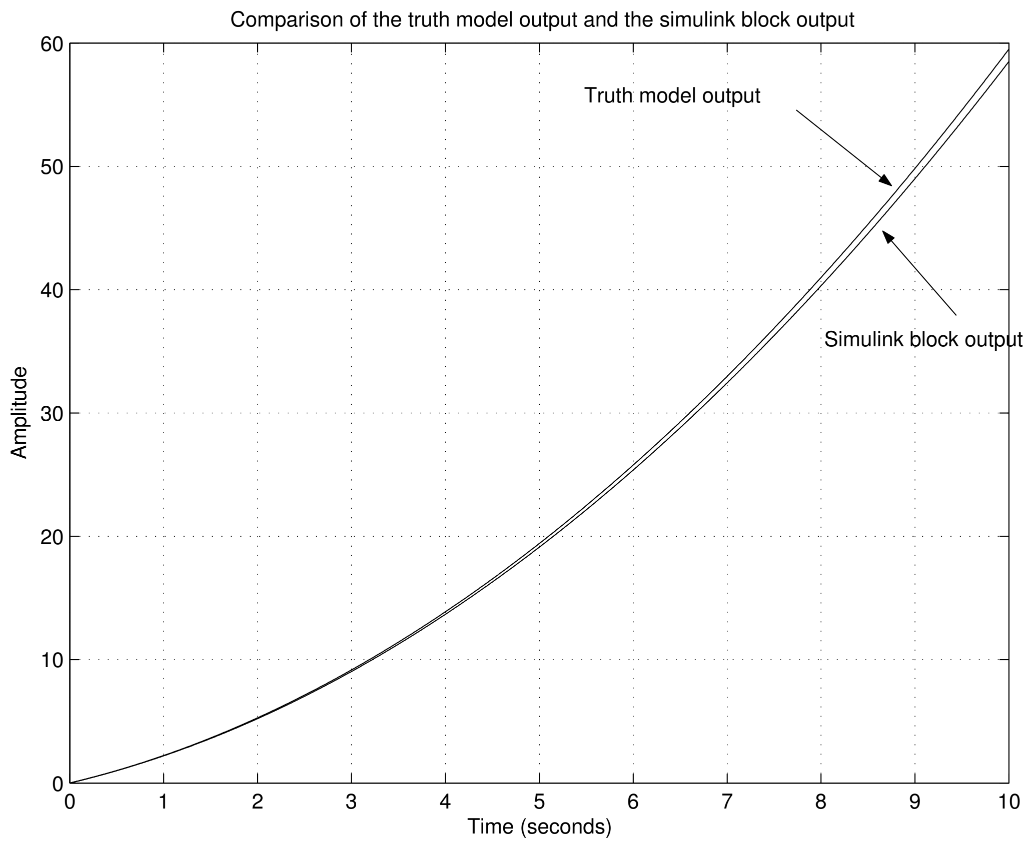

Fig. 10. Comparison of the truth model output and the SIMULINK block output for the polynomial input.

not proved to be as simple as the linear delays. A first order transfer function model are obtained to be a satisfactory model. Table II shows some of the transfer function approximations for a few cases of linear delay and sine input. Following observations are made from a number of simulation results with sinusoidal input shown in table II.

1. As the frequency of the input signal increases, the approximation model poles move towards the origin.

2. Zeros of the approximation model, remain unchanged with the change in the

Table II. Variation in the approximation model parameter with change in the frequency of sinusoidal inputs.

Input	Delay	Transfer Function
$sin(2\pi 0.001t)$	$0.1t$	$\frac{0.3947s+78.94}{s+86.85}$
$sin(2\pi 0.002t)$	$0.1t$	$\frac{0.4519s+90.38}{s+99.38}$
$sin(2\pi 0.003t)$	$0.1t$	$\frac{0.4309s+86.18}{s+94.70}$
$sin(2\pi 0.004t)$	$0.1t$	$\frac{0.3910s+78.20}{s+85.87}$
$sin(2\pi 0.005t)$	$0.1t$	$\frac{0.3065s+61.30}{s+67.23}$
$sin(2\pi 0.006t)$	$0.1t$	$\frac{0.2411s+48.23}{s+52.82}$
$sin(2\pi 0.007t)$	$0.1t$	$\frac{0.1866s+37.33}{s+40.81}$
$sin(2\pi 0.008t)$	$0.1t$	$\frac{0.1467s+29.33}{s+32.01}$
$sin(2\pi 0.009t)$	$0.1t$	$\frac{0.1179s+23.58}{s+25.67}$
$sin(2\pi 0.01t)$	$0.1t$	$\frac{0.09592s+19.18}{s+20.83}$
$sin(2\pi 0.02t)$	$0.1t$	$\frac{0.02516s+5.033}{s+5.264}$
$sin(2\pi 0.03t)$	$0.1t$	$\frac{0.01274s+2.549}{s+2.543}$
$sin(2\pi 0.04t)$	$0.1t$	$\frac{0.008714s+1.743}{s+1.671}$
$sin(2\pi 0.05t)$	$0.1t$	$\frac{0.007164s+1.433}{s+1.345}$
$sin(2\pi 0.06t)$	$0.1t$	$\frac{0.006742s+1.348}{s+1.266}$
$sin(2\pi 0.07t)$	$0.1t$	$\frac{0.007099s+1.42}{s+1.347}$
$sin(2\pi 0.08t)$	$0.1t$	$\frac{0.008018s+1.604}{s+1.526}$
$sin(2\pi 0.09t)$	$0.1t$	$\frac{0.008445s+1.689}{s+1.571}$
$sin(2\pi 0.1t)$	$0.1t$	$\frac{0.007787s+1.557}{s+1.403}$

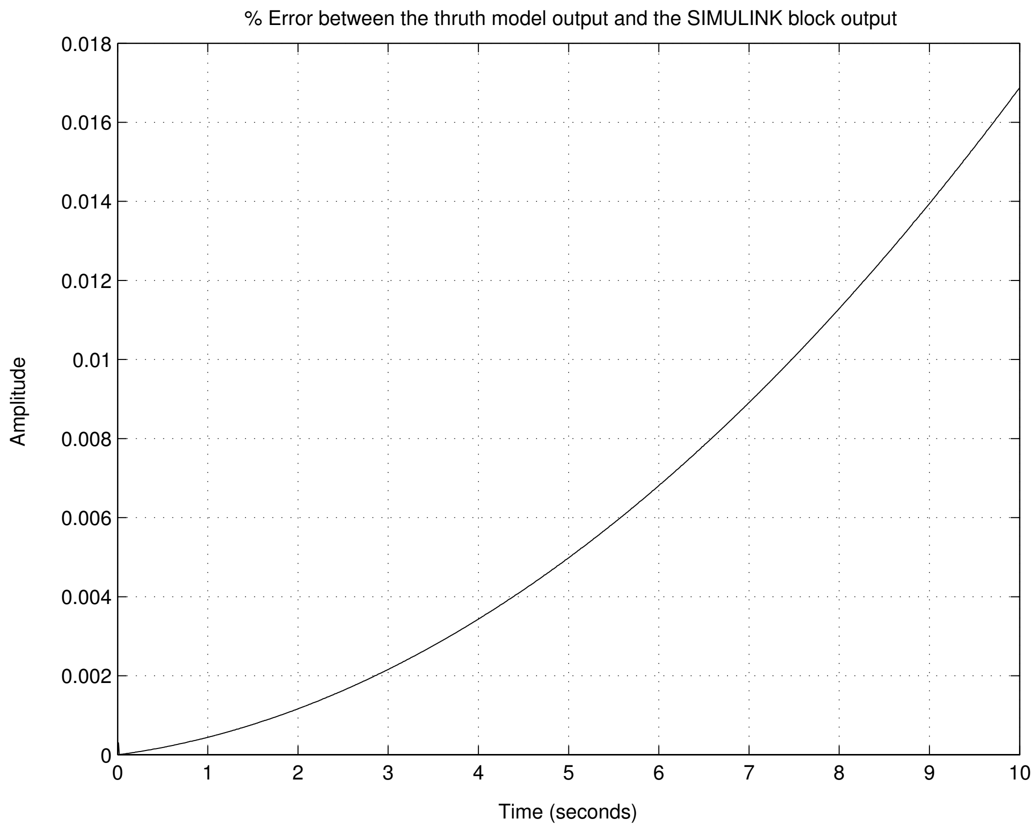

Fig. 11. Percentage error between the truth model output and the SIMULINK block output for the polynomial input.

input frequency.

Table III shows how change in the delay affects the approximation model. Following observations were made from a number of simulation results with sinusoidal input shown in table III.

1. As the first derivative of delay signal increases, the approximation poles move toward origin. The model is not valid for delay derivative that are greater than one.

2. Zeros of the approximation model, remain unchanged with the change in the

Table III. Variation in the approximation model parameters with change in the linearly time-varying delays.

Input	Delay	Transfer Function
$sin(2\pi 0.01t)$	$0.01t$	$\frac{0.5540s+110.8}{s+111.8}$
$sin(2\pi 0.01t)$	$0.02t$	$\frac{0.3822s+76.45}{s+77.76}$
$sin(2\pi 0.01t)$	$0.03t$	$\frac{0.2848s+56.96}{s+58.44}$
$sin(2\pi 0.01t)$	$0.04t$	$\frac{0.2247s+44.93}{s+43.49}$
$sin(2\pi 0.01t)$	$0.05t$	$\frac{0.1843s+36.87}{s+38.46}$
$sin(2\pi 0.01t)$	$0.06t$	$\frac{0.1560s+31.20}{s+32.81}$
$sin(2\pi 0.01t)$	$0.07t$	$\frac{0.1350s+26.99}{s+28.62}$
$sin(2\pi 0.01t)$	$0.08t$	$\frac{0.1188s+23.77}{s+25.40}$
$sin(2\pi 0.01t)$	$0.09t$	$\frac{0.1061s+21.22}{s+22.86}$
$sin(2\pi 0.01t)$	$0.1t$	$\frac{0.09592s+19.18}{s+20.83}$
$sin(2\pi 0.01t)$	$0.2t$	$\frac{0.0493s+9.859}{s+11.55}$
$sin(2\pi 0.01t)$	$0.3t$	$\frac{0.03376s+6.752}{s+8.499}$
$sin(2\pi 0.01t)$	$0.4t$	$\frac{0.02595s+5.191}{s+6.988}$
$sin(2\pi 0.01t)$	$0.5t$	$\frac{0.02127s+4.253}{s+6.1}$
$sin(2\pi 0.01t)$	$0.6t$	$\frac{0.01814s+3.628}{s+5.525}$
$sin(2\pi 0.01t)$	$0.7t$	$\frac{0.01585s+3.171}{s+5.109}$
$sin(2\pi 0.01t)$	$0.8t$	$\frac{0.01415s+2.83}{s+4.814}$
$sin(2\pi 0.01t)$	$0.9t$	$\frac{0.01278s+2.556}{s+4.575}$

input frequency.

Let us consider this example of sinusoidal input which is used to show results graphically.

$$\begin{aligned} u(t) &= sin(2\pi 0.1t), \\ \tau(t) &= 0.1t. \end{aligned}$$

$$y(t+0.1t) = sin(2\pi 0.1t). \tag{3.8}$$

Input sinusoid has frequency of 0.1 Hz. When this sinusoid is subjected to the linearly time-varying delay, the output behaviour obtained is as shown in figure 12. System identification of this input-output data is carried out and following transfer function is obtained.

$$\frac{y(s)}{u(s)} = \frac{0.007787s + 1.557}{s + 1.403}.$$

A first order model structure is obtained as a satisfactory approximation for this case. Output obtained with this approximation is compared with the truth model output as shown in figure 13. Figure 14 shows the % error between the truth model output and the approximation output.

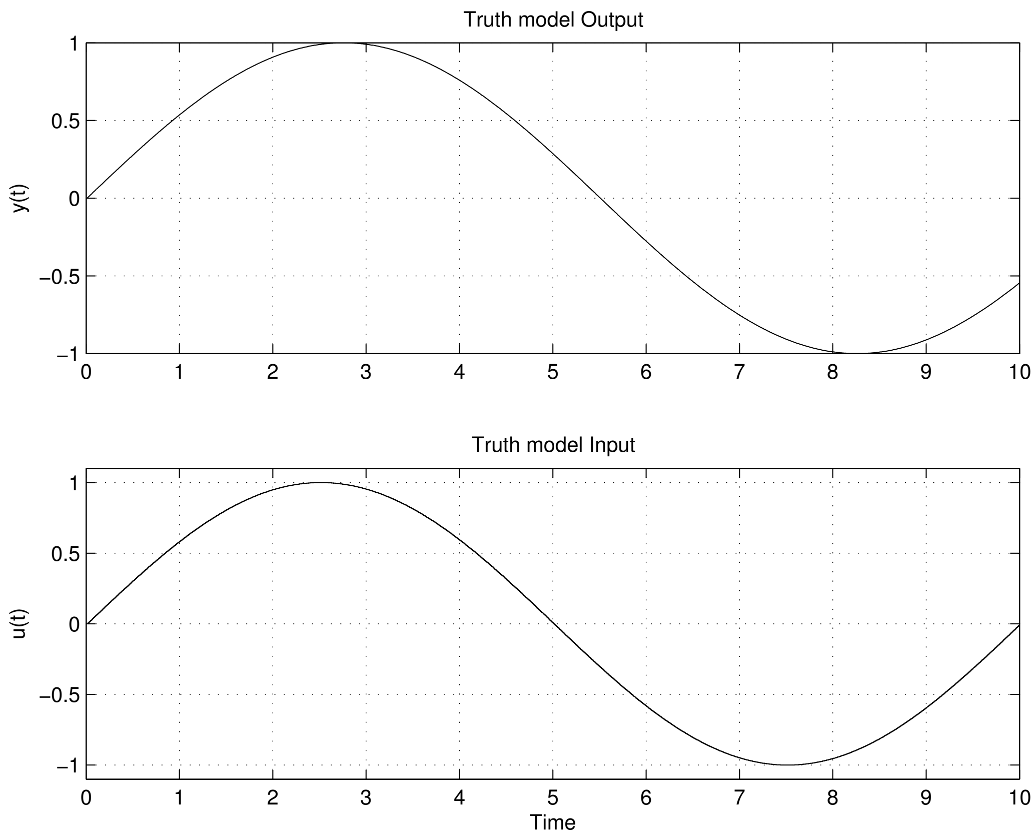

Fig. 12. Input-output behaviour for the truth model for the sinusoidal input with linearly time-varying delay.

B. Sinusoidal time-varying delays

Systems with sinusoidal time-varying delay, is analyzed for ramp, polynomial and sinusoidal inputs. Sinusoidal delay is defined as

$$\tau(t) = |sin(2\pi f t)| \tag{3.9}$$

where f is the frequency in Hz and $\tau(t) \geq 0$. As delay can never be negative, restriction is imposed on the sine function. Also $(2\pi f) < 1$ so that the first delay

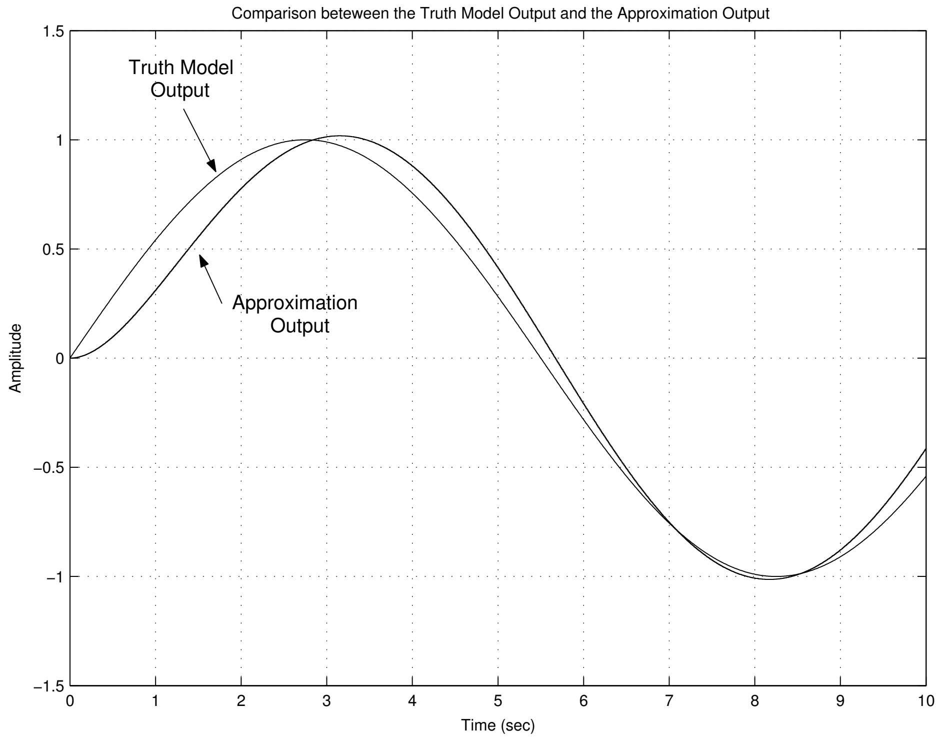

Fig. 13. Comparison of the truth model output and the approximation output for the sinusoidal input with linearly time-varying delay.

derivative is always restricted to be less than one. The truth model output is obtained for sinusoidal time-varying delay. Satisfactory first order approximation models are obtained in this case. Different inputs considered are as follows.

1. Ramp input

The truth model output for the following ramp input and sinusoidal time-varying delay is obtained.

$$\begin{aligned} u(t) &= t, \\ \tau(t) &= |sin(2\pi 0.03t)|. \end{aligned}$$

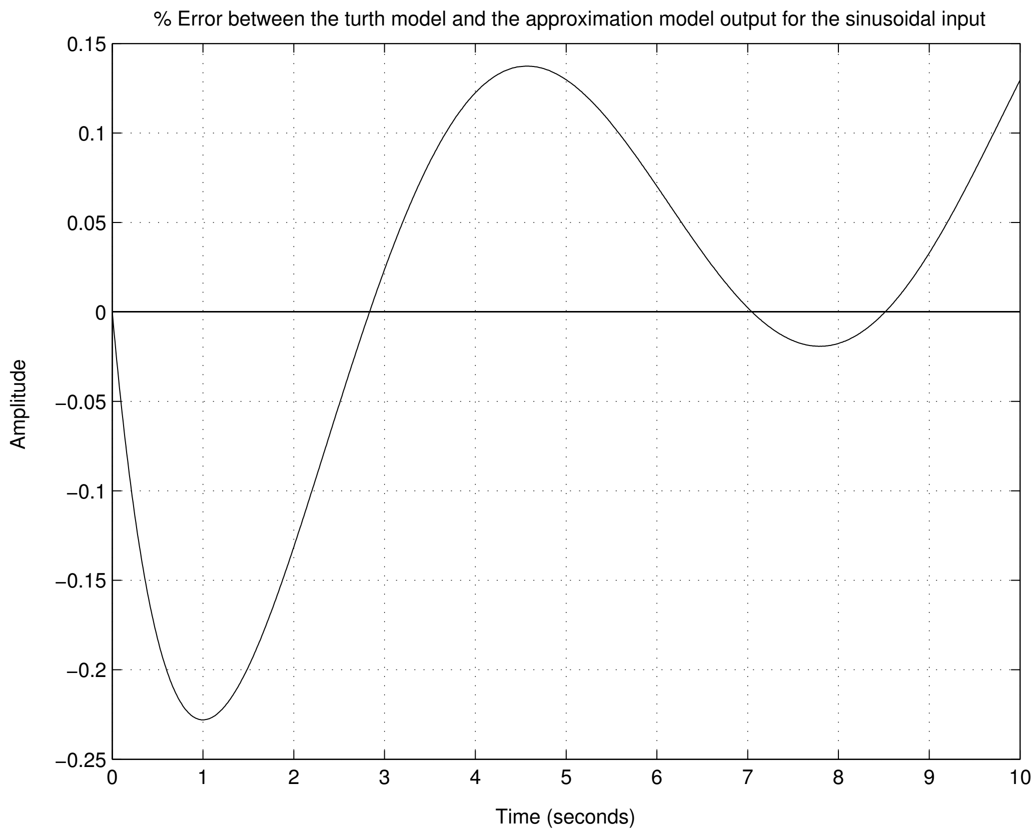

Fig. 14. Percentage error between the truth model output and approximation output for the sinusoidal input with linearly time-varying delay.

$$y(t + |sin(2\pi 0.03t)|) = t. \tag{3.10}$$

Figure 15 shows input-output response for the truth model for the ramp input. Approximation model obtained for the above input-output data is given by

$$\frac{y(s)}{u(s)} = \frac{0.01334s + 2.67}{s + 2.89}.$$

Figure 16 shows the comparison between the truth model output and the approximation model output for the ramp input. Figure 17 shows the error between the truth

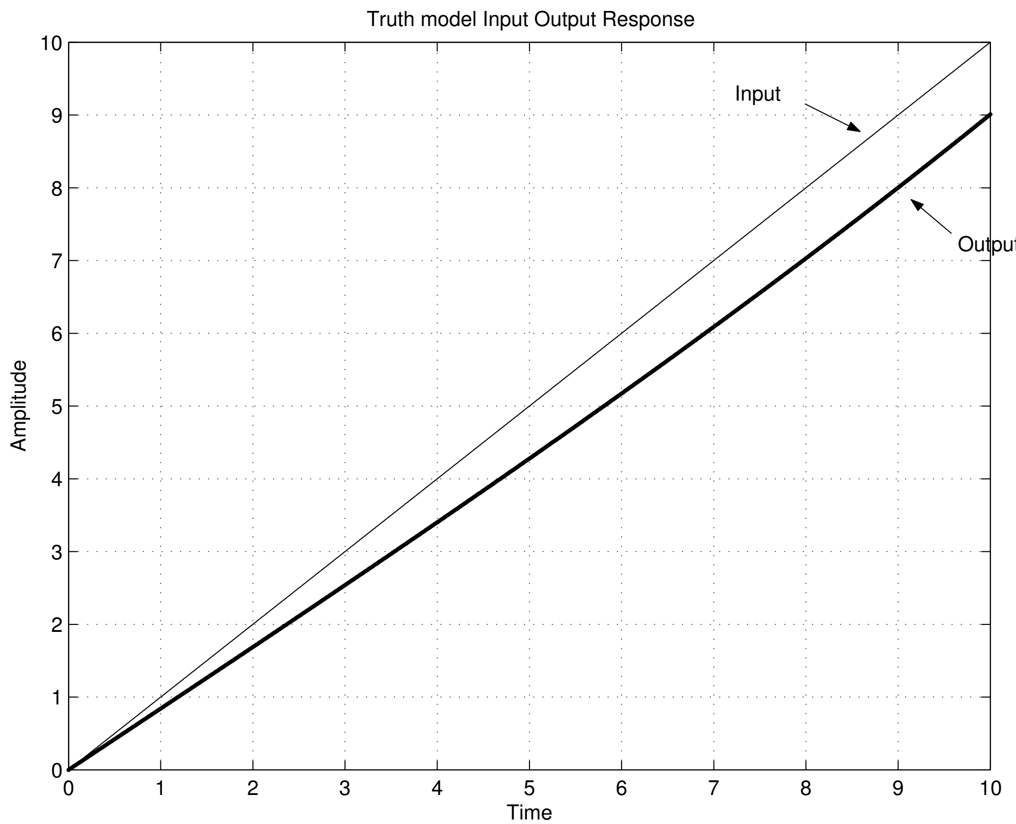

Fig. 15. Input-output response for the truth model for ramp input with sinusoidal time-varying delay.

model output and the approximation model output for ramp input.

The truth model output is obtained for the ramp input by changing the frequency of the sinusoidal time-varying delay. Postulated approximation models are then simulated to observe the trends with changing parameter of the delay. Table IV shows the changes in the approximation model parameters for different frequency of the delay.

Following observations are made from a number of simulation results with a sinusoidal delay.

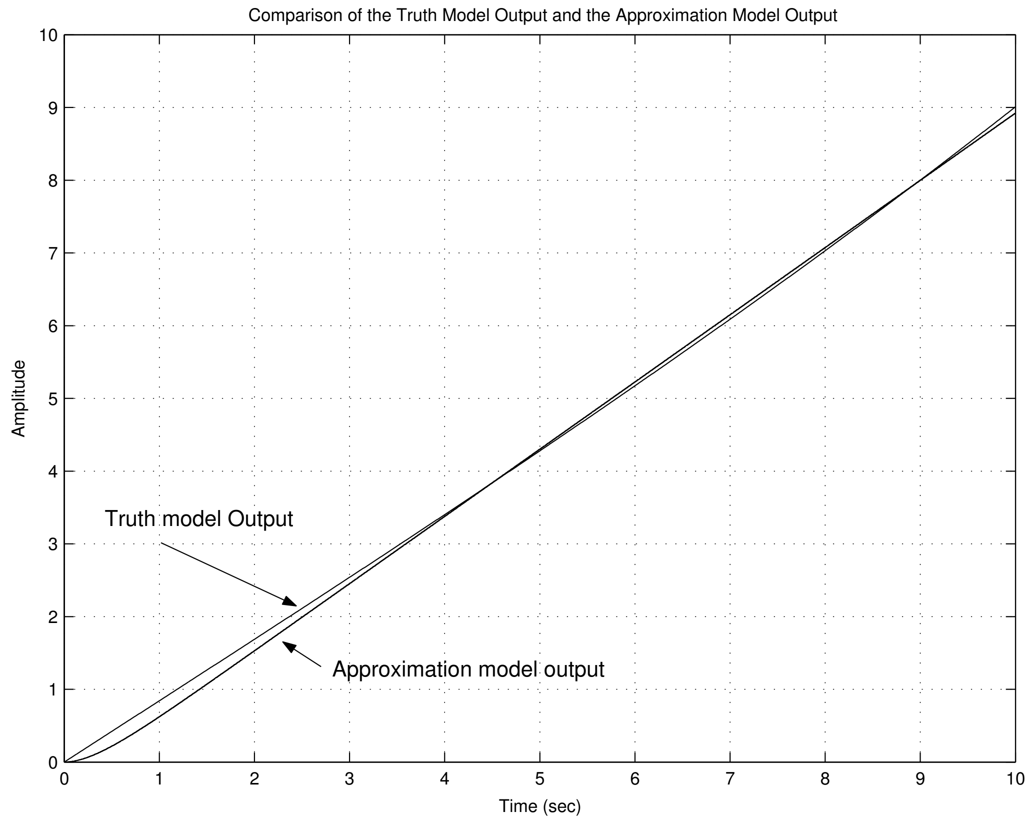

Fig. 16. Comparison between the truth model output and the approximation model output for the ramp input with sinusoidal time-varying delay.

1. As the frequency of the delay signal increases, the approximation model poles move toward origin.

2. Zeros of the approximation model, remain unchanged with the change in the frequency.

Similarly, table V lists approximation models obtained for a degree two polynomial as the input function.

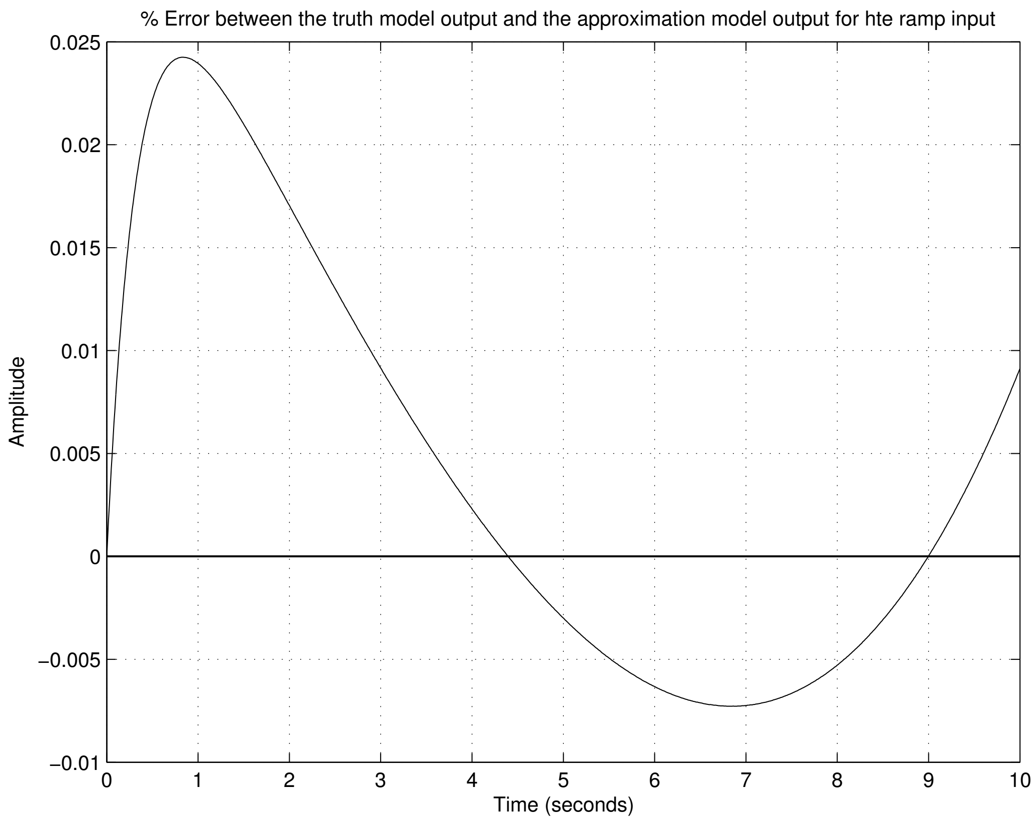

Fig. 17. Percentage error between the truth model output and the approximation model output for the ramp input with sinusoidal time-varying delay.

2. Sinusoidal input

The truth model output for the following sinusoidal input and the sinusoidal time-varying delay is obtained.

$$\begin{aligned} u(t) &= sin(2\pi 0.1t), \\ \tau(t) &= |sin(2\pi 0.04t)|. \end{aligned}$$

$$y(t + |sin(2\pi 0.04t)|) = sin(2\pi 0.1t). \tag{3.11}$$

Figure 18 shows the truth model input and output behaviour for sinusoidal input.

Table IV. Variation in the approximation model parameters with change in frequency of the the sinusoidal delay signal.

Input	Delay	Transfer Function
t	$sin(2\pi 0.006t)$	$\frac{0.7059s+141.2}{s+146.4}$
t	$sin(2\pi 0.007t)$	$\frac{0.5937s+118.7}{s+123.8}$
t	$sin(2\pi 0.008t)$	$\frac{0.4842s+96.85}{s+101.5}$
t	$sin(2\pi 0.009t)$	$\frac{0.3870s+77.40}{s+81.57}$
t	$sin(2\pi 0.010t)$	$\frac{0.3064s+61.28}{s+64.89}$
t	$sin(2\pi 0.020t)$	$\frac{0.04431s+8.86}{s+9.69}$
t	$sin(2\pi 0.030t)$	$\frac{0.01334s+2.67}{s+2.89}$
t	$sin(2\pi 0.040t)$	$\frac{0.00536s+1.07}{s+1.06}$
t	$sin(2\pi 0.050t)$	$\frac{0.00258s+0.51}{s+0.39}$

Approximation model obtained in this case is,

$$\frac{y(s)}{u(s)} = \frac{0.0059s + 1.195}{s + 1.028}.$$

Figure 19 shows the comparison between the truth model output and the approximation model output for a sinusoidal input. Figure 20 shows the % error between the truth model output and the approximation model output for the sinusoidal input. Approximation models are obtained for various sinusoidal inputs. Table VI shows a few calculations for sinusoidal inputs.

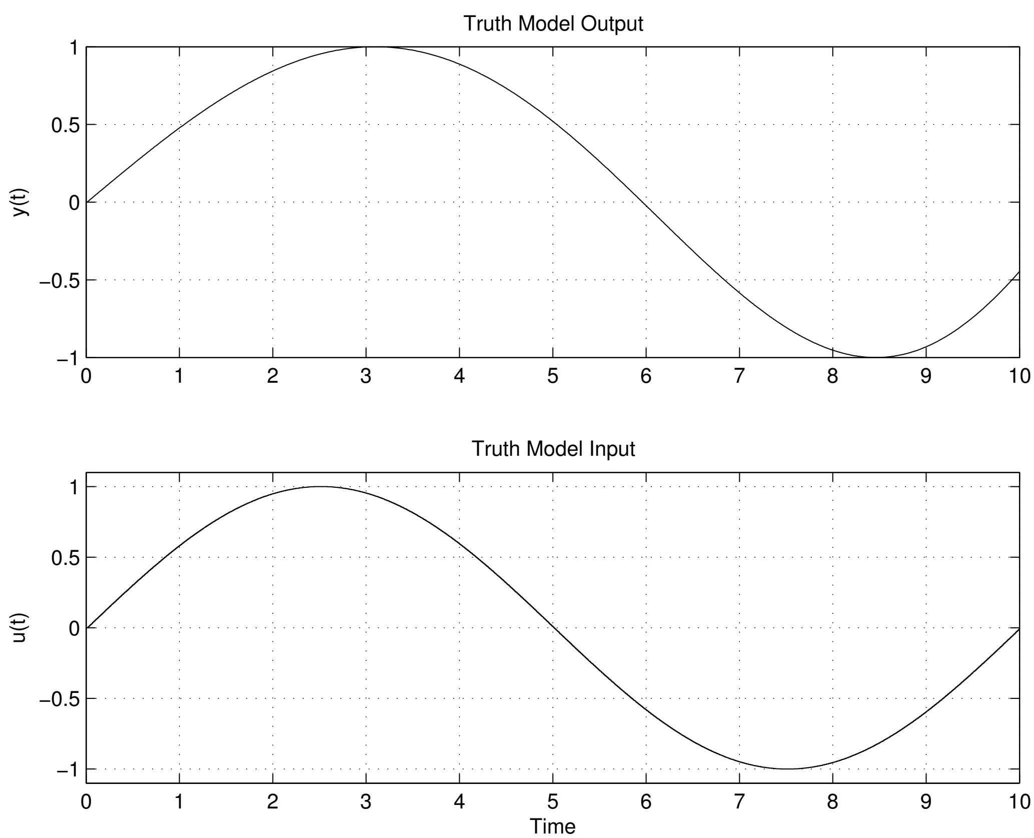

Fig. 18. Input-output graph for the sinusoidal input with sinusoidal time-varying delay.

Table V. Variation in the approximation model parameter with change in frequency of the sinusoidal delay in the case of a degree two polynomial as the input function.

Input	Delay	Transfer Function
t^2	$sin(2\pi 0.006t)$	$\frac{0.4657s+93.13}{s+100}$
t^2	$sin(2\pi 0.007t)$	$\frac{0.3525s+70.49}{s+76.54}$
t^2	$sin(2\pi 0.008t)$	$\frac{0.2655s+53.10}{s+58.25}$
t^2	$sin(2\pi 0.009t)$	$\frac{0.2010s+40.21}{s+44.53}$
t^2	$sin(2\pi 0.010t)$	$\frac{0.1541s+30.82}{s+34.44}$
t^2	$sin(2\pi 0.020t)$	$\frac{0.0223s+4.46}{s+5.218}$
t^2	$sin(2\pi 0.030t)$	$\frac{0.0067s+1.33}{s+1.44}$
t^2	$sin(2\pi 0.040t)$	$\frac{0.0026s+0.52}{s+0.36}$

Table VI. Variation in the approximation model parameters with change in frequency of the input sinusoid and the delay function.

Input	Delay	Transfer Function
$sin(2\pi 0.006t)$	$sin(2\pi 0.006t)$	$\frac{0.3725s+74.50}{s+77.11}$
$sin(2\pi 0.007t)$	$sin(2\pi 0.007t)$	$\frac{0.2741s+54.81}{s+57.00}$
$sin(2\pi 0.008t)$	$sin(2\pi 0.008t)$	$\frac{0.1984s+39.68}{s+41.42}$
$sin(2\pi 0.009t)$	$sin(2\pi 0.009t)$	$\frac{0.1460s+29.20}{s+30.58}$
$sin(2\pi 0.010t)$	$sin(2\pi 0.010t)$	$\frac{0.1092s+21.83}{s+22.93}$
$sin(2\pi 0.020t)$	$sin(2\pi 0.020t)$	$\frac{0.0175s+3.50}{s+3.65}$
$sin(2\pi 0.030t)$	$sin(2\pi 0.030t)$	$\frac{0.0082s+1.63}{s+1.63}$

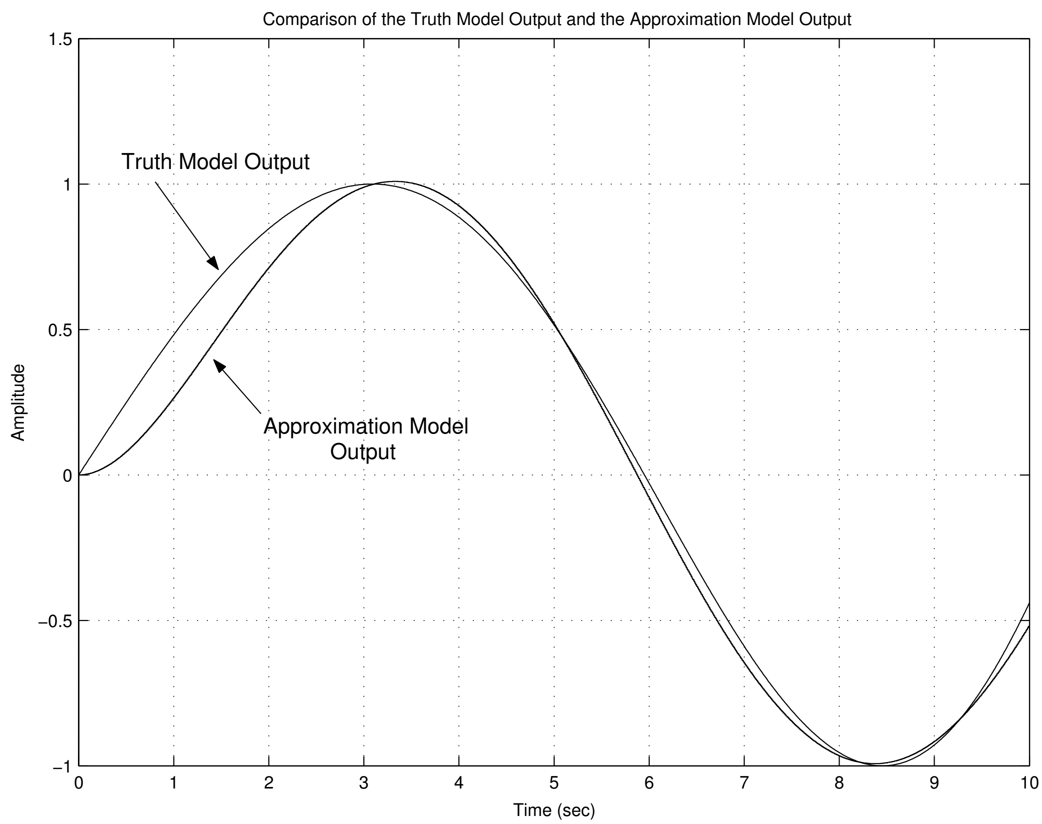

Fig. 19. Comparison between the truth model output and the approximation model output for sinusoidal input with sinusoidal time-varying delay.

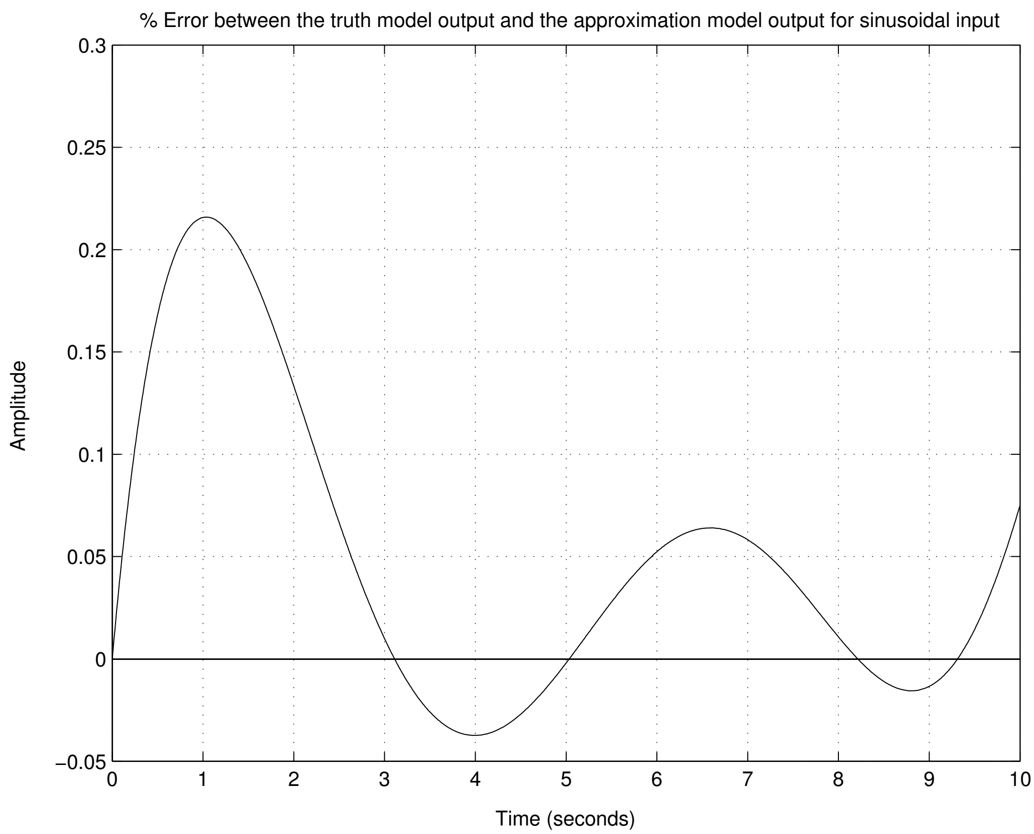

Fig. 20. Percentage error between the truth model output and the approximation model output for the sinusoidal input with sinusoidal time-varying delay.

CHAPTER IV

SUMMARY AND CONCLUSIONS

This thesis presents approximation models for time-varying time delays. A method is developed to obtain the true output of systems with time-varying delay. System identification techniques are employed to obtain approximations.

A. Summary

The problem is stated and objectives are set in Chapter I. Relevant literature is reviewed as an introduction to current research in the area of time delay.

A relationship between the input function, output function and time-varying time delay is defined in Chapter II. A method to obtain the behaviour of the system that has time-varying time delay is developed on the basis of this relationship. To validate this method, limit analysis is performed with the known results for fixed delay. Obtained results validate the methodology.

Approximation models are developed in Chapter III. Two types of time-varying delays considered in this research are

1. Linearly time-varying delay.

2. Sinusoidally time-varying delay.

Step, ramp, polynomial and sinusoidal input functions are considered for study. It is observed that the model structure is dependent on the class of inputs. Hence for different class of inputs, different approximation models are obtained. Satisfactory approximation model obtained in case of linearly time-varying delay. Attempt is made to develop generalized approximation model for sinusoidally time-varying delays. Results obtained with the approximation models are compared with the truth model output.

B. Conclusions

The approximations obtained are a good fit to the truth model output in case of ramp inputs. Errors between the approximation model output and the truth model output are of the order of 0.01 % in the case of ramp inputs. In the case of general polynomial inputs, it has been observed that the structure of parameters depends on the first derivative of the delay and the degree of the input polynomial function. Approximation error increases with higher degree polynomial. With sinusoidal delays, a first order approximation model is obtained. In this case, the obtained models are dependent on the frequency of the delay and are in good fit for the lower frequency delays. It has been observed that sinusoidal delay brings more frequency components into the system. A generalization of the approximations obtained in the case of sinusoidal delays is not easily achievable.

The truth model outputs are compared with the outputs obtained with the SIMULINK "Variable Transportation Delay" block. The SIMULINK block considers time-varying delay like a simple transport lag with time dependent characteristics. Error in the SIMULINK block output indicates that relation it uses for time-varying delay is not valid.

Some conclusions that can be made based on the results of this research include:

1. A generalized model is obtained in the case of linearly time-varying delay subjected to polynomial, ramp and step inputs.

2. Approximation model structure in the case of time-varying delay is found to be dependent on the first derivative of delay as well as the input function parameters.

3. MATLAB SIMULINK “Variable Transportation Delay” block, which uses simple transportation lag is not the true representation for time-varying delay.

C. Future work

This work focuses on two types of the time-varying delays. This work need to be extended to various types of time-varying delays. Approximation models can be obtained for exponentially decaying delays, delay in steps, discrete delays, periodic delays, random delays, Network delays, etc. Currently, varying delays are approximated as fixed delays to solve control problems. Application of these dynamic models in these problems will bring results with better fit. These approximation models should be applied in process industries, internet networks and robotics applications. Also issues related to the stability and performance of these approximation models can be studied.

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VITA

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